INVESTIGATION OF LWR MODEL WITH FLUX FUNCTION DRIVEN BY RANDOM FREE FLOW SPEED

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**OUTLINE**

- Motivation of the study
  - interesting empirical observations
- Formulation of a new LWR model
  - incorporating the randomness
- Illustrative example of application
  - evaluation of predictability of traffic evolution
- Concluding remarks
First and second order speed-density relation at one typical site based on GA400 ITS data
Motivation

- Two types of randomness
  - statistical: ubiquitous and controllable
  - inherent: unique, reflecting the system dynamics

- Reflection of the LWR model
  - to account for the observed randomness
  - new insight into the operations of transportation system
FORMULATION: RETROSPECT

• Conservation law:

\[
\int_{\Omega_x} k(x, t) \, dx + \int_{\Omega_t} f(k(x, t)) \, dt = 0
\]

\[
f(k(x, t)) = k(x, t)v(k(x, t))
\]
FORMULATION: $k$-$v$ CURVE

- Curve $\gamma : (k, v)$, parameterized by $0 \leq k \leq k_j$

$$\left( \frac{v(k)}{v_f} \right)^\alpha + \left( \frac{k}{k_j} \right)^\beta = 1$$

where $\alpha, \beta > 0$, interpolates $(0, v_f)$ and $(k_j, 0)$
FORMULATION: INTERPOLATION

- Brownian bridge

\[ B^0(t) = B(t)\left|\begin{array}{l} B(0) = 0, \ B(1) = 0 \end{array}\right. \]

- A generalization

\[ V^0(k) = V(k)\left|\begin{array}{l} V(s) = \tilde{V}_s, \ s \in \mathbb{I} \end{array}\right. \]
FORMULATION: random $v_f$

- Define $v_f$ a random process

$$v_f(k, \omega) : (\mathbb{R}^+, \Omega) \mapsto \mathbb{R}^+$$

In particular, assume

$$v_f(k, \omega) \overset{d}{\sim} \bar{v}_f + (sk + r)\epsilon$$

The flux function driven by $v_f$

$$f(k) \overset{d}{=} k(\bar{v}_f + (sk + r)\epsilon)(1 - \left(\frac{k}{k_j}\right)^\beta)^{1/\alpha}$$
**Formulation: Moments**

- Moment properties

\[
E(v(k)) = \bar{\nu}_f \left(1 - \left(\frac{k}{k_j}\right)^\beta\right)^{1/\alpha}
\]

\[
Var(v(k)) = (sk + r)^2 \left(1 - \left(\frac{k}{k_j}\right)^\beta\right)^{2/\alpha}
\]
FORMULATION: DECOMPOSITION

- Smoothness
- Boundedness

\[
\sup_{0 \leq k \leq k_j} |f'(k)| \leq 5k_j |\epsilon_s| + 3|\epsilon_r| + 3\bar{\nu}_f
\]

- Decomposition

\[
f(k) = f^+(k) + f^-(k)
\]
EXAMPLE: RANDOM FLUX FUNCTION

Hypothetical random speed-density relation (left panel) with $\alpha = 1$, $\beta = 1$, $s = 0.05$ and $r = 3$ and 100 realizations of flux function (right panel)
EXAMPLE: ENO-FD SCHEME

3rd order ENO (essentially non-oscillatory)-FD (finite difference)

Temporal development of density $k$ with $\epsilon = 0$: rarefaction waves (left panel) with $(k_l, k_r) = (110, 30)$, and shock waves (right panel) with $(k_l, k_r) = (30, 110)$
EXAMPLE: SKETCH OF ALGORITHM

- Sketch of algorithm
  1. Generate $\epsilon$, and obtain corresponding random flux function;
  2. Solve the LWR model using the ENO-FD;
  3. Go back to 1.
Propagation of local disturbance (jam: left panel; vacuum: right panel) with random flux function, $t = 600$ sec, 20 realizations
EXAMPLE: IMPLICATION

- Quantities of interest

\[ k_* = \max_{0 \leq x_i \leq 10} |k(x_i) - 50|, \quad x_* = \arg\max_{x_i} |k(x_i) - 50| \]

- Coefficient of variation (CoV = \(\sigma/\mu\))

<table>
<thead>
<tr>
<th></th>
<th>Mean of (k_*)</th>
<th>Std of (k_*)</th>
<th>Mean of (x_*)</th>
<th>Std of (x_*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>condition a</td>
<td>16.09</td>
<td>0.50</td>
<td>5.99</td>
<td>0.75</td>
</tr>
<tr>
<td>condition b</td>
<td>26.46</td>
<td>1.92</td>
<td>8.63</td>
<td>0.63</td>
</tr>
</tbody>
</table>

\[ CoV_{a,k_*} > CoV_{b,k_*}, \quad CoV_{a,x_*} > CoV_{b,x_*} \]
CONCLUDING REMARKS

• **Summary**
  - Faithfulness, well-posedness and ease of solution
  - Randomness in the scope

• **Gaps**
  - Estimation
  - Temporal and spatial inhomogeneity