

Existence, stability, and mitigation of gridlock in beltway networks

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Abstract

Previous studies have shown that the gridlock state can arise in a beltway network. However, no closed-form formulations were provided for traffic dynamics in a beltway network, and the existence and stability of gridlock were not established systematically. In this study we first present the network kinematic wave model for traffic dynamics in a rotationally symmetric beltway network, including the merging and diverging models. Then we demonstrate that gridlock is always a stationary solution with fixed exiting ratios. We further show that the gridlock state may be stable or unstable, depending on the merging priority and the exiting ratio. Furthermore, we discuss the mechanisms and limitations of existing mitigation strategies and propose a new adaptive driving strategy. Finally we conclude with future research directions.

Keywords: Beltway network; Gridlock; Exiting ratios; Merging priority; Stationary states; Stability; Evacuation diverging strategy.

1. Introduction

A beltway network is a ring road with multiple entrance and exit links, as shown in Figure 1, which can be embedded in both freeway and arterial networks. A gridlock state in a beltway network is defined as a traffic state when the total density is positive, but the flow-rate is zero. A trivial case is when all links, including the entrance and exit links, are totally jammed, and no vehicles can move around. This can occur when all exit links are blocked. However, more interesting gridlock states can arise when all entrance links are jammed but all exit links are completely empty. The latter type of non-trivial gridlock states are concerned in this study.

In (Daganzo, 1996), it was shown that a non-trivial gridlock state can appear in a beltway network, and the sufficient condition for the occurrence of gridlock in such networks was identified with macroscopic merging and diverging rules in the Cell Transmission Model (Daganzo, 1995). In (Daganzo, 2007), the occurrence of gridlock was also demonstrated in an

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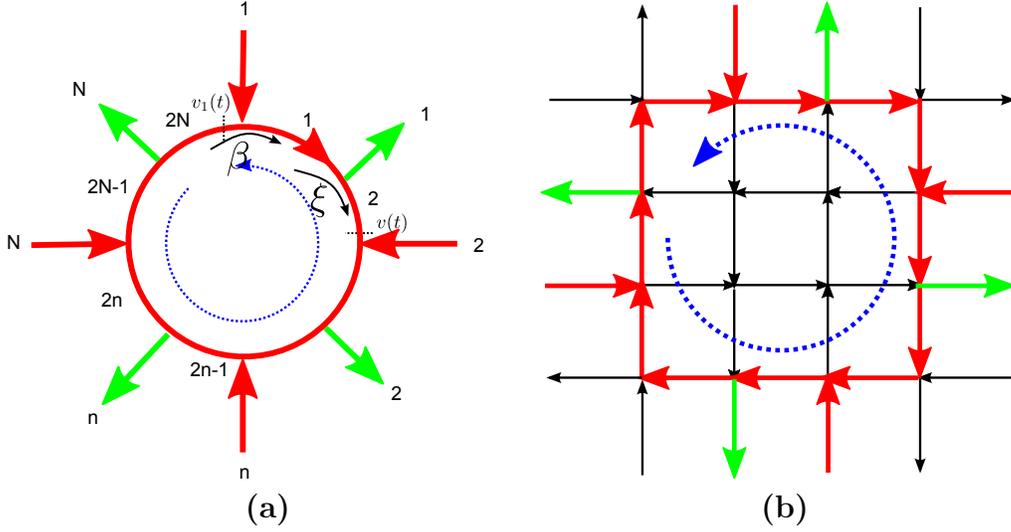


Figure 1: Beltways embedded in (a) a freeway network and (b) an arterial grid network

urban beltway network with continuous entrance and exit links. Further in (Daganzo, 2011), the dynamics related to the occurrence and mitigation of gridlock were studied in double-ring networks, and the impacts of drivers' adaptation to traffic conditions were analyzed. In (Jin et al., 2013), it was shown that a double-ring network can reach a gridlock state at multiple density levels in the kinematic wave model. In (Jin, 2013b), it was shown that the occurrence of a gridlock state is associated with circular information propagation on a ring road, and that a gridlock state is a fixed point of a Poincaré map. In the literature, however, there lacks a systematic treatment on existence, stability, and mitigation of gridlock in a beltway network.

In this study, we attempt to fill the gap. First we apply the network kinematic wave model, in which the LWR model is used to describe traffic dynamics on each link, and macroscopic merging and diverging rules are used to prescribe traffic dynamics at merging and diverging junctions. Then we mathematically prove that, in a traffic statics problem with constant demands at the entrance links and constant supplies at the exit links, there exist stationary states in the network, and gridlock is always a stationary solution. Furthermore, with the Poincaré map developed in (Jin, 2013b), we show that the gridlock state may be stable or unstable, depending on relationship between the merging priority and the diverging ratio. Finally, we propose a feedback control algorithm to mitigate gridlock and discuss its effect on the existence and stability of gridlock.

The rest of the paper is organized as follows. In Section 2, we present the network kinematic wave model for traffic dynamics in the beltway network. In Section 3, we formulate the traffic statics problem in the beltway network and prove the existence of gridlock stationary states. In Section 4, we apply the Poincaré map to study the stability of the gridlock state. In Section 5, we study the impacts of an evacuation diverging model on the

gridlock state. In Section 6, we conclude with discussions and possible future studies.

2. Network kinematic wave model

For a rotationally symmetric beltway network, shown in Figure 1(a), N entrances (or on-ramps) and N exits (or off-ramps) ($N \geq 2$) are labeled from 1 to N , respectively. We assume that all mainline links between an entrance and an exit have the same length, L , and label them from 1 to $2N$ in a clockwise fashion. In addition, we label the merging junctions by $2n - 1$ and the diverging junctions by $2n$ ($n = 1, \dots, N$).

On mainline link a ($a = 1, \dots, 2N$), at a point (a, x_a) and time t , traffic density, speed, and flow-rate are denoted by $k_a(x_a, t)$, $v_a(x_a, t)$, and $q_a(x_a, t)$, respectively. We assume that all links share the same fundamental diagram (Greenshields, 1935): $q = Q(k)$, which is unimodal and attains its capacity $C = Q(K_c)$ at the critical density of K_c . An example is the following triangular fundamental diagram:

$$Q(k_a) = \min\{Vk_a, W(K - k_a)\}, \quad (1)$$

where V is the free-flow speed, K the jam density, and $-W$ the shock wave speed in congested traffic. For a flow-rate q_a , $K_1(q_a)$ and $K_2(q_a)$ are the respective under-critical (UC) and over-critical (OC) densities for q_a ; i.e., $Q(K_1(q_a)) = Q(K_2(q_a)) = q_a$, and $K_1(q_a) \leq K_c \leq K_2(q_a)$. The triangular fundamental diagram and $K_1(\cdot)$ and $K_2(\cdot)$ are shown in Figure 2.

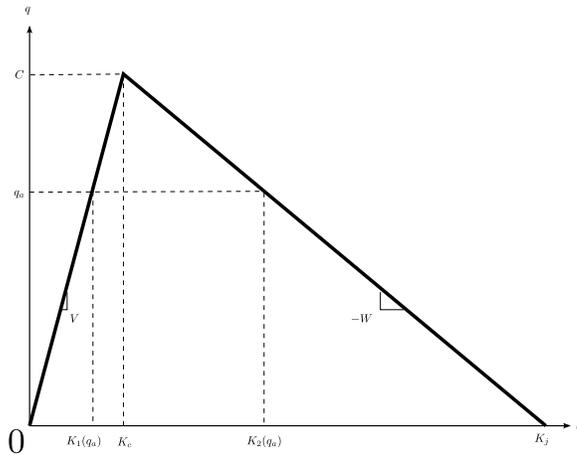


Figure 2: The triangular fundamental diagram and flow-density relation

Traffic dynamics on link a , either a mainline, entrance, or exit link, are described by the LWR model (Lighthill and Whitham, 1955; Richards, 1956):

$$\frac{\partial k_a}{\partial t} + \frac{\partial Q(k_a)}{\partial x_a} = 0, \quad (2)$$

in which traffic queues propagate and dissipate with shock waves. In the celebrated Cell Transmission Model (Daganzo, 1995; Lebacque, 1996), the LWR model is extended for traffic dynamics in a network in the following two steps. First, the following traffic demand and supply functions are defined:

$$d = D(k_a) \equiv Q(\min\{K_c, k_a\}), \quad (3a)$$

$$s = S(k_a) \equiv Q(\max\{K_c, k_a\}). \quad (3b)$$

Second, traffic dynamics at the merging and diverging junctions can be described by the following macroscopic models:

1. At merging junction $2n - 1$, the boundary fluxes are given by

$$q_{2n-1}(0, t) = \min\{d_{2n-2}(L^-, t) + \delta_n(t), s_{2n-1}(0^+, t)\}, \quad (4a)$$

$$q_{2n-2}(L, t) = \min\{d_{2n-2}(L^-, t), \max\{s_{2n-1}(0^+, t) - \delta_n(t), \beta s_{2n-1}(0^+, t)\}\}, \quad (4b)$$

where $2n - 2 = 2N$ when $n = 1$, $q_{2n-1}(0, t)$ is the in-flux of link $2n - 1$, $q_{2n-2}(L, t)$ the out-flux of link $2n - 2$, $d_{2n-2}(L^-, t)$ the downstream demand of link $2n - 2$, $s_{2n-1}(0^+, t)$ the upstream supply of link $2n - 1$, and $\delta_n(t)$ the demand of entrance n . Here $1 - \beta$ is the merging ratio of the entrance, and β that of the mainline link. This is the priority-based merging model proposed in (Daganzo, 1995), which was shown to be invariant in (Jin, 2010) and thus can be used to analyze traffic dynamics in the network without worrying about the existence of so-called interior states.

2. At diverging junction $2n$, the boundary fluxes are given by

$$q_{2n-1}(L, t) = \min\{d_{2n-1}(L^-, t), \frac{s_{2n}(0^+, t)}{\xi}, \frac{\sigma_n(t)}{1 - \xi}\}, \quad (5a)$$

$$q_{2n}(0, t) = \xi q_{2n-1}(L, t), \quad (5b)$$

where $q_{2n-1}(L, t)$ is the out-flux of link $2n - 1$, $q_{2n}(0, t)$ the in-flux of link $2n$, $d_{2n-1}(L^-, t)$ the downstream demand of link $2n - 1$, $s_{2n}(0^+, t)$ the upstream supply of link $2n$, and $\sigma_n(t)$ the supply of exit n . Here $1 - \xi$ is the diverging proportion to the exit, and ξ that to the mainline link. This is the first-in-first-out (FIFO) diverging model proposed in (Daganzo, 1995) and was shown to be invariant in (Jin, 2013a). Note that here all vehicles are assumed to have predefined destinations in the diverge model (5).

Thus (2), (4), and (5) form the network kinematic wave model, which is the continuous version of CTM, for traffic dynamics in the beletway network. It can be considered a control system, where the traffic densities, $k_a(x_a, t)$, are state variables, the entrance demands, $\delta_n(t)$, and exit supplies, $\sigma_n(t)$, are inputs, and the merging priority, β , and the diverging ratio, ξ , are control variables. Note that $k_a(x_a, t)$ is of an infinite dimension, and the control system is infinite-dimensional and cannot be easily analyzed.

3. Traffic statics problem and existence of gridlock

Following (Jin, 2012), we define the traffic statics problem for the beltway network as finding stationary solutions to the kinematic wave model, (2) with (4) and (5), when the origin demands and destination supplies are all constant; i.e., $\delta_n(t) = \delta$ and $\sigma_n(t) = \sigma$.

3.1. Stationary traffic conditions on links and at junctions

As shown in (Jin, 2012; Jin et al., 2013), when link a becomes stationary, its density can be written as ($x_a \in [0, L]$)

$$k_a(x_a, t) = H(u_a L - x_a)K_1(q_a) + (1 - H(u_a L - x_a))K_2(q_a), \quad (6)$$

where $u_a \in [0, 1]$ is the uncongested fraction of the road, $H(\cdot)$ is the Heaviside function,

The upstream supply and the downstream demand are both constant: $s_a(0^+, t) = s_a$, and $d_a(L^-, t) = d_a$. Thus there are four types of stationary states:

$$\text{Strictly under-critical (SUC): } u_a = 1, q_a < C, d_a = q_a, \text{ and } s_a = C \quad (7a)$$

$$\text{Strictly over-critical (SOC): } u_a = 0, q_a < C, d_a = C, \text{ and } s_a = q_a \quad (7b)$$

$$\text{Critical (C): } u_a \in [0, 1], q_a = C, d_a = C, \text{ and } s_a = C \quad (7c)$$

$$\text{Zero-speed shock wave (ZS): } u_a \in (0, 1), q_a < C, d_a = C, \text{ and } s_a = C \quad (7d)$$

A stationary state on link a is illustrated in Figure 3. Note that all types of stationary states can be considered as special cases of a zero-speed shock wave.

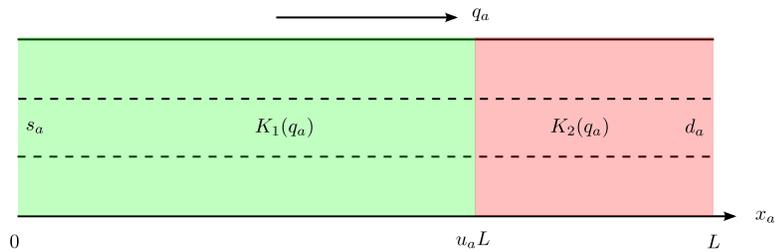


Figure 3: Stationary states on a link

In stationary states around merging and diverging junctions, (4) and (5) can be simplified as follows.

1. At merging junction $2n - 1$,

$$q_{2n-1} = \min\{d_{2n-2} + \delta, s_{2n-1}\}, \quad (8a)$$

$$q_{2n-2} = \min\{d_{2n-2}, \max\{s_{2n-1} - \delta, \beta s_{2n-1}\}\}. \quad (8b)$$

2. At diverging junction $2n$,

$$q_{2n-1} = \min\left\{d_{2n-1}, \frac{s_{2n}}{\xi}, \frac{\sigma}{1-\xi}\right\}, \quad (9a)$$

$$q_{2n} = \xi q_{2n-1}. \quad (9b)$$

Therefore, to solve the traffic statics problem, we need to find q_a , u_a , d_a , and s_a for $a = 1, \dots, 2N$ from the system of algebraic equations in (7), (8), and (9). In general, since each link can have four possible types of stationary states, there can be 4^{2N} possible combinations, and a brute force method as in (Jin, 2012) cannot apply in this case.

3.2. Existence of gridlock and other stationary solutions

In the gridlock state in the beltway network, we have $q_a = s_a = 0$, $u_a = 1$, and $d_a = C$ for $a = 1, \dots, 2N$.

Lemma 3.1. *If $\delta > 0$ and $\sigma > 0$, then the entrance links are jammed, and the exit links are empty.*

Proof. For entrance link n at merging junction $2n - 1$, since the flow-rates on links $2n - 1$ and $2n - 2$ are both 0, the flow-rate on the entrance link is also 0. Thus its upstream supply, s , has to be 0, since, otherwise, its upstream flow-rate equals $\min\{\delta, s\} > 0$. From (7) we can see that a link whose upstream supply is 0 has to be totally jammed.

For exit link n at diverging junction $2n$, its flow-rate is also 0. Thus its downstream supply, d , has to be 0, since, otherwise, its downstream flow-rate equals $\min\{d, \sigma\} > 0$. From (7) we can see that a link whose upstream supply is 0 has to be empty. ■

Theorem 3.2. *In a beltway network, the gridlock state is always a stationary solution.*

Proof. First, clearly the gridlock state is a stationary state on a link as defined in (7). Second, at merging junction $2n - 1$, since $d_{2n-2} = C$ and $s_{2n-1} = 0$, we have

$$\begin{aligned} q_{2n-1} &= 0 = \min\{C + \delta, 0\}, \\ q_{2n-2} &= 0 = \min\{C, \max\{-\delta, 0\}\}. \end{aligned}$$

Thus the gridlock state satisfies the merge model. Third, at diverging junction $2n$, since $d_{2n-1} = C$ and $s_{2n} = 0$, we have

$$\begin{aligned} q_{2n-1} &= 0 = \min\left\{C, 0, \frac{\sigma}{1-\xi}\right\}, \\ q_{2n} &= 0 = \xi q_{2n-1}. \end{aligned}$$

Thus the gridlock state satisfies the diverge model. Therefore the gridlock state is a stationary solution of the network kinematic wave model with constant entrance demand and exit supply. ■

Theorem 3.2 is a very important observation, even though the mathematical proof it is quite straightforward. From this theorem we can conclude that, if a beltway network is initially jammed, it will always be jammed. Given Lemma 3.1, this is a bit surprising, since there are sufficient spaces at the exit links and sufficient vehicles at the entrance links. The occurrence of such a gridlock state is highly related to the diverging model, in which all vehicles have predefined routes, even though there are better options in other routes. This suggests that route guidance strategies can be helpful for preventing the occurrence of gridlock in a beltway network.

4. Stability of gridlock

In the gridlock state, all mainline links are congested, and traffic waves propagate upstream. Thus we have a counter-clockwise circular information propagation path along the ring road. As shown in Figure 1, we define two Poincaré sections at the upstream points of two consecutive merging junctions 3 and 1, and denote the two out-fluxes of the mainline road by $v(t) = q_2(L, t)$ and $v_1(t) = q_{2N}(L, t)$ respectively.

After T_1 , $v(t)$ propagates to the diverging junction 2, and $s_2(0^+, t + T_1) = v(t)$, since link 2 is congested. From (5) we have

$$q_1(L, t + T_1) = \frac{v(t)}{\xi}.$$

After T_2 , $q_1(L, t + T_1)$ propagates to the merging junction 1, and $s_1(0^+, t + T_1 + T_2) = q_1(L, t + T_1)$, since link 1 is congested. From (4) we have

$$q_{2N}(L, t + T) = \beta q_1(L, t + T_1) = \frac{\beta}{\xi} v(t),$$

where $T = T_1 + T_2$. Then after N pairs of entrance and exit links, we obtain the following Poincaré map:

$$v(t + NT) = \left(\frac{\beta}{\xi}\right)^N v(t). \quad (10)$$

Note that the Poincaré map is only valid when all links are congested and cannot capture other types of stationary states.

For the Poincaré map, (10), the gridlock state $v(t) = 0$ is always an equilibrium point. Furthermore, from the Poincaré map, we can determine the stability of the gridlock state: (i) When $\frac{\beta}{\xi} < 1$, the gridlock state is asymptotically stable, and the half-life equals $\frac{\ln 2}{-N \ln(\beta/\xi)}$. (ii) When $\frac{\beta}{\xi} > 1$, the gridlock state is unstable. (iii) When $\frac{\beta}{\xi} = 1$, there can exist multiple Lyapunov stable stationary states. It can be verified that these results are consistent with those in (Daganzo, 1996).

5. Mitigation of gridlock with adaptive driving

5.1. Existing mitigation strategies

For a beltway network described by (2) with (4) and (5), when the exit links are uncongested, three types of control strategies have been discussed in (Daganzo, 1996, 2007, 2011) to mitigate the impacts of gridlock:

1. The demand $\delta_n(t)$ can be regulated by ramp metering. This may delay the occurrence of gridlock. But when all links are congested, if $\frac{\beta}{\xi} < 1$, the system will converge to the gridlock state irreversibly.
2. The merging ratio for the entrances, $1 - \beta$, can be decreased by signal settings at the intersections. This can destabilize the gridlock state when $\frac{\beta}{\xi} > 1$.
3. The diverging proportion to the exits, $1 - \xi$, can be increased by variable message signs and adaptive driving behaviors. This can also destabilize the gridlock state when $\frac{\beta}{\xi} > 1$.

These results can be explained by (10). However, as shown in Section 3, these strategies may not avoid the occurrence of gridlock, since the gridlock state is always a stationary solution to the system.

5.2. A new adaptive driving strategy

In this subsection, we consider another type of adaptive driving strategy, derived from the evacuation model in (Jin, 2013a). At a diverge $2n$, we apply the following diverging model

$$q_{2n-1}(L, t) = \min\{d_{2n-1}(L^-, t), s_{2n}(0^+, t) + \sigma_n(t)\}, \quad (11a)$$

$$q_{2n}(0, t) = \min\{s_{2n}(0^+, t), \max\{d_{2n-1}(L^-, t) - \sigma_n(t), \xi d_{2n-1}(L^-, t)\}\}, \quad (11b)$$

where $\xi \in [0, 1]$ is the priority for adaptive drivers to choose the ring road. In a sense, this is a feedback control strategy, as the diverging ratio is determined by the dynamic traffic conditions.

Here we just consider symmetric stationary solutions on the ring road, in which all odd-numbered links share the same traffic conditions, denoted by q_1 , u_1 , d_1 , and s_1 , and all even-numbered mainline links shared the same traffic conditions, denoted by q_2 , u_2 , d_2 , and s_2 . Thus from (8) and (11) we have

$$q_1 = \min\{d_2 + \delta, s_1\}, \quad (12a)$$

$$q_2 = \min\{d_2, \max\{s_1 - \delta, \beta s_1\}\}, \quad (12b)$$

$$q_1 = \min\{d_1, s_2 + \sigma\}, \quad (12c)$$

$$q_2 = \min\{s_2, \max\{d_1 - \sigma, \xi d_1\}\}. \quad (12d)$$

Lemma 5.1. *When $\delta > 0$ and $\sigma > 0$, the gridlock state is no longer a solution of (12).*

Proof. We assume that the gridlock state is still a stationary solution of (12); i.e., $d_2 = d_1 = C$, and $s_1 = s_2 = 0$. From the first two equations, we can see that $q_1 = q_2 = 0$. However, from the third equation, $0 = \min\{C, \sigma\} > 0$, which is impossible. Thus the gridlock state is no longer a solution. ■

Note that the adaptive driving strategy yields the same results as without using it under uncongested conditions. In this case, $q_1 = d_1 \leq s_1 = C$, and from (12c) and (12d) we have $q_1 = d_1$ and $q_2 = \xi d_1$. That is, vehicles can still use their original routes under uncongested conditions. However, when traffic is congested, some vehicles will have to be re-routed to avoid the development of gridlock.

6. Conclusion

In this study, we formulated the traffic dynamics in a beltway network with the network kinematic wave model incorporating the LWR model as well as invariant merging and diverging models. We then studied the stationary states, including the gridlock state, and their stability. Finally we discussed existing mitigation strategies and proposed a new evacuation-based strategy, which eliminates the gridlock state in stationary states.

The closed-form network kinematic wave model of the traffic dynamics in the beltway network have enabled the analyses of stationary states and their stability properties. In this study, however, the discussions are at the macroscopic level. In the future we will be interested in finding the optimal strategy that prevents gridlock but involves the minimum number of re-routings at the microscopic level. In addition, we will be interested in designing the best mitigation strategies that can lead to the minimum delays. We will also be interested in designing the adaptive control strategies for mitigating traffic congestion and avoiding gridlock in more general networks.

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