STOCHASTIC APPROXIMATIONS FOR THE MFD OF URBAN NETWORKS

Jorge Laval
Felipe Castrillón
Yi Zhou
MOTIVATION

Urban congestion is high

- U.S. yearly delay per auto commuter: 38 hours (2012)
- 11 countries with higher INRIX congestion index (Europe and N America)
- Increasing urbanization and motor vehicle ownership

Transportation demand models

- O-D table is expensive and requires too many inputs
- Results are highly sensitive to OD table leading to high inaccuracies

MACROSCOPIC FUNDAMENTAL DIAGRAM

- “Well-defined” relationship of vehicle accumulation and output flow
- Insensitive to O-D demands and route choice (?)
- Invariant between days and time of day
- Tool for macroscopic feedback monitoring and control
METHOD OF CUTS FOR HOMOGENEOUS CORRIDORS

\[ q(k) = \inf_s \{ ks + R(s) \} \]
Define $N(t,x)$ as the vehicle counts at location $x$ before time $t$.

The Fundamental Diagram is a Hamilton-Jacobi equation:

$$\frac{dN}{dt} = Q\left(-\frac{dN(t,x)}{dx}, t, x\right)$$


VARIATIONAL THEORY

Hopf-Lax solution: \[ N_P = \inf_{B \in B_P} \{ N_B + \Delta_{BP} \} \]

MFD

\[ q(k) = \inf_s \{ ks + R(s) \} \]
$q(k) = \inf_S \{ks + R(s)\}$
APPROXIMATING VALID PATHS

Heterogeneous network
RENEWAL REWARDS EXAMPLE

Renewal process is counting process with times between successive events i.i.d with distribution D

Example:

Renewal Process: Bus $n$ has headway (interarrival time) $Y_n \sim D_1$
Reward: Number of passenger that alight is $X_n \sim D_2$

Flow of passengers:

$$\text{CLT: } \lim_{t \to \infty} \frac{X(t)}{t} \sim N(\mu, \sigma^2/t) \text{ with:}$$

$$\mu = \frac{\mu_X}{\mu_Y}$$
$$\sigma^2 = \frac{\mu_X^2}{\mu_Y} \left( \delta_X^2 + \delta_Y^2 - 2 \text{cor}(X, Y) \delta_Y \delta_X \right),$$
variables

\( Y_i^S \) is the renewal cycle

\( X_i^S \) is the reward

\( X_B^S = \sum X_i^S \)

“cut” strategies

- \( s_0 \): stay in the same intersection (stationary observer),
- \( s_1^- \): travel at speed \( w^- \) and stop when hitting a red phase,
- \( s_2^- \): stop one intersection before \( s_1^- \).
KEY RESULT

MFD \implies q(k) = \min_s \{q_s(k)\},

\quad q_s(k) = \lim_{t \to \infty} \frac{1}{t} X_s^B(t),

Theorem 1. (Distribution of cuts). The distribution of $q_s(k)$ is Normal with mean $\mu_s$ and variance $\sigma_s^2/t$, where:

\begin{align*}
\mu_s &= \frac{\mu_X}{\mu_Y} \\
\sigma_s^2 &= \frac{\mu_X^2}{\mu_Y} \left( \delta_X^2 + \delta_Y^2 - 2 \text{cor}(X, Y) \delta_Y \delta_X \right).
\end{align*}
DIMENSIONLESS FORMULATION

Reduce the formulation to the minimum number of parameters

Three transformations:
1) fundamental diagram
2) remaining parameters
3) density transformation
 DIMENSIONLESS VARIABLES

$$\theta = \frac{w^\#}{w^b}$$

$$Q = 1, \quad \kappa = 1,$$

$$\rho \equiv \frac{\mu_r}{\mu_g} \quad \text{and} \quad \lambda \equiv \frac{\mu_\ell}{\mu_g}$$

![Diagram showing flow vs. density with points w^# and -w^b]
DENSITY TRANSFORMATION
DISTRIBUTION OF CUTS

Back to Theorem 1...

mean:

\[
\begin{align*}
\mu_{s_0} &= \frac{1}{1 + \rho}, \\
\mu_{s_1}(k') &= \frac{1 \pm 2k'}{1 + c_1}, \quad \text{where: } c_1 \equiv c\rho/(1 + \rho), c \equiv (1 + \delta^2)\rho/\lambda, \\
\mu_{s_2}(k') &= \frac{1 + c \pm 2k'}{1 + c + 2\rho^2/\lambda}.
\end{align*}
\]

COV\(^2\):

\[
\begin{align*}
\delta_{s_0}^2 &= \frac{\mu_g 2\delta^2 \rho^2}{T \rho + 1}, \\
\delta_{s_1}^2 &= \frac{\mu_g (1 + \delta^2)^2 \rho^3 (1 + \delta^2 (1 + 2\rho))}{T 2(1 + \rho) ((1 + \delta^2) \rho^2 + (1 + \rho)\lambda)}, \\
\delta_{s_2}^2(k') &= \frac{\mu_g}{T} f(\lambda, \rho, \delta, \pm k'),
\end{align*}
\]

MFD (distribution of min):

\[
F_{q(k')}(q) = 1 - \prod_{s \in \Omega} \left(1 - F_{s,k'}(q)\right),
\]
DISTRIBUTION OF CUTS

Back to Theorem 1...

mean:

\[ f(\lambda, \rho, \delta) \]

COV\(^2\):

\[ f(\lambda, \rho, \delta) \]

MFD (distribution of min):

\[ F_{q(k')}(q) = 1 - \prod_{s \in \Omega} (1 - F_{s,k'}(q)) \]
TRAFFIC SIMULATION EXPERIMENTS

Lattice implementation of KW model with triangular FD with discretized values for vehicle number $\Delta n$ and time $\Delta t$, with dimensionless variables $\tilde{t} = t/\Delta t$ and $\tilde{n} = n/\Delta n$

Exact solution:

$$X_{\tilde{t},\tilde{n}} = \min \left\{ X_{\tilde{t}-1,\tilde{n}} + \theta \Delta x, X_{\tilde{t}-1,\tilde{n}-1} - \Delta x \right\}$$

1) Generate random corridor (5000 corridor):
   - $w = 20$ km/hr
   - $\kappa = 150$ veh/km
   - $\theta = 4$
   - $\mu_1 = 0.2$ km,
   - #signals = 15
   - simulation time = 15 min
2) Initial value problem with constant density $k$
3) Ring-road

J. Laval, L. Leclercq. The Hamilton-Jacobi partial differential equation and the three representations of traffic flow. Transportation Research Part B.
SIMULATION RESULTS LONG BLOCKS
YOKOHAMA DATA
YOKOHAMA VS. MODEL

The graph shows a comparison between Yokohama data and a model. The 90th percentile line indicates the upper bound of the model's predictions, while the 10th percentile line shows the lower bound. The scatter plot represents individual data points from Yokohama, with the model fitting closely to the data.
CONCLUSION

• Formulation with only three parameters: \( \lambda, \rho, \delta \)

• Stochastic MFD symmetry

• Main finding: good fit with Yokohama data indicates that

\[
\text{Network MFD} = \text{Corridor MFD}(\bar{\lambda}, \bar{\rho}, \delta)
\]
DISTRIBUTION OF CUTS

\[ X = (1 \pm 2k')L/2 + G, \]
\[ Y = L/2 + G + R, \]

\[ \mu_X = (1 \pm 2k')\mu_L/2 + \mu_G, \quad \mu_Y = \mu_L/2 + \mu_G + \mu_R, \]
\[ \sigma_X^2 = (1 \pm 2k')^2\sigma_L^2/4 + \sigma_G^2, \quad \sigma_Y^2 = \sigma_L^2/4 + \sigma_G^2 + \sigma_R^2, \]
\[ \text{Cov}(X, Y) = (1 \pm 2k')\sigma_L^2/4 + \sigma_G^2. \]

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